Electromagnetic Diffusion into a Moving Conductor

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An exact solution is obtained for electromagnetic diffusion into a moving slab of finite electrical conductivity. A magnetic field is suddenly established on one side of the initially field-free slab. At some distance from the other side, a stationary plane of infinite conductivity acts as a terminus to the fields. The mathematical problem is posed in terms of the vector potential that satisfies the ordinary diffusion equation in the reference frame of the moving fluid. The velocity, which implies motional induction, appears in the boundary conditions. The solution is obtained by Laplace transform techniques. A dimensionless parameter appears whose magnitude is that of the magnetic Reynolds number of hydromagnetic theory. It is seen here that the character of the exponential terms in the solution, which reflect the location of the poles of the inversion integral, depend on the sign of this parameter, i.e., the direction of the velocity, as well as its magnitude.

I. Introduction

THE diffusion of a suddenly established magnetic field into or through a stationary conductor has been calculated for various cases.[1-3] The present paper is concerned with the corresponding calculation for a moving conductor that can be regarded either as a solid or as an incompressible fluid. This calculation is of interest not only in its own right but also because it yields a precise mathematical description of the combined effects of magnetic diffusion and magnetic transport under various conditions, together with an algebraic parameter whose absolute value can be identified with the magnetic Reynolds number of magnetohydrodynamics.[4]

We consider a plane nonmagnetic conducting slab of infinite extent, moving at constant velocity $\mathbf{v}$ in the normal direction. At time zero, a uniform tangential magnetic field is turned on in the region on one side of the slab and is then held constant in that region. It is desired to calculate the electric and magnetic fields in the interior of the slab as functions of position and time. The formulation will be based on Maxwell's equations, with the displacement current neglected (see Ref. 1, Chap. XI) and with zero charge density. We shall use potentials, with zero scalar potential and with div $\mathbf{A} = 0$. We shall have to carry out a Lorentz transformation from the laboratory system to the rest system of the slab, and we shall assume that $v^2 \ll c^2$, so that this reduces to a Galilean transformation. We use mks units.

In Sec. 2 the problem is briefly discussed for the case $v = 0$. In Sec. 3 the general problem is formulated, and a Laplace transformation is applied. Section 4 deals with the mathematical details of the inversion of the Laplace transformation, and Sec. 5 is a discussion of results.

2. Magnetic Diffusion through a Stationary Plane Slab

Let the slab, of conductivity σ and of magnetic permittivity equal to μ_0 extend from $z = 0$ to $z = l$, as in Fig. 1. Let the external magnetic field $\mathbf{H}_0$, in the x direction, be turned on at $t = 0$ and held constant in time and uniform throughout region 0 for $t > 0$.

We can choose $\mathbf{A}$ to be in the $\pm y$ direction. In region 0, $\mathbf{A}$ satisfies the vector Laplace equation, and, therefore, for $t > 0$,

$$A_{0y} = -\mu_0 z H_{0x} + C(t) \qquad (1)$$

where $C(t)$ is yet to be determined. In region 2, the magnetic field must be uniform and in the x direction but can vary in time. In order to make the problem mathematically determinate, there must be a boundary condition on $\mathbf{A}_2$ at the upper boundary of region 2 (which may be either at a finite value of z or at $z = \infty$); we shall provide such a boundary condition by assuming a perfectly conducting plane sheet at $z = b$; then we have

$$A_{2y} = -\mu_0(z - b)H_{2x}(t) \qquad (2)$$

where $H_{2x}(t)$ is yet to be determined.

Maxwell's equations require that $\mathbf{A}$ and $\mathbf{H}$ be continuous at $z = 0$ and at $z = l$; by combining these conditions, one can get the boundary conditions on A_{1y}:

$$[\partial A_{1y}/\partial z]_{z=0} = -\mu_0 H_{0x} \qquad (3)$$

$$(b - l)[\partial A_{1y}/\partial z]_{z=l} + A_{1y}(l, t) = 0 \qquad (4)$$

Also, A_{1y} satisfies the diffusion equation

$$\partial^2 A_{1y}/\partial z^2 = \mu_0\sigma(\partial A_{1y}/\partial t) \qquad (5)$$

and the initial condition

$$A_{1y}(z, 0) = 0 \qquad (6)$$

The system (3–6) has been solved[5] for the analogous heat-conduction problem of a slab with zero initial temperature, constant heat flux into one face, and radiation from the other face into a medium at zero temperature. The solution is

$$A_{1y}(z, t) = -\mu_0 H_0(z - b) - 2l\mu_0 H_0 \sum_{n=1}^{\infty} \frac{(\zeta_n^2 + K^2)\cos(\zeta_n z/l)}{\zeta_n^2(K + K^2 + \zeta_n^2)} e^{-\zeta_n^2(t/\tau)} \qquad (7)$$

where $\tau = \mu_0\sigma l^2$, $K = l/b - l)$, and ζ_n is the nth positive root of the transcendental equation,

$$\zeta \tan\zeta = K \qquad (8)$$

all of whose roots are real and have been tabulated (see the Appendix of Ref. 5).

One can now find A_{0y} and A_{2y} by using Eqs. (1) and (2) and the continuity of A_y at $z = 0$ and $z = l$.

3. Magnetic Diffusion through a Uniformly Moving Plane Slab

Let the slab be moving at constant velocity v relative to the laboratory system, with v either positive or negative

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according to whether the motion is in the $\pm z$ direction. We use unprimed symbols to refer to the laboratory system and primed symbols for the slab-rest system. Let the fixed and moving origins coincide at $t = 0$, as shown in Fig. 2. Then the transformation of coordinates, time, and potentials from the laboratory system to the slab system is given by the equations (with $v^2 \ll c^2$ and with the unprimed scalar potential equal to zero)

$$x' = x \qquad y' = y \qquad t' = t \qquad z' = z - vt \tag{9}$$

$$A_x' = A_x = 0 \qquad A_y' = A_y \qquad A_z' = A_z = 0 \qquad \varphi' = 0 \tag{10}$$

It may be noted here that, if v were not constant, all of these equations would still be valid in the approximation used here (ie., $v^2 \ll c^2$), except that the last of Eqs. (9) would have to be replaced by

$$z' = z - \int_0^t v dt \tag{11}$$

Equations (1) and (2) must still be valid. By using Eqs. (9) and (10), we can transform them to the slab system, getting

$$A_{0y}' = -\mu_0(z' + vt')H_{0x}' + C(t') \tag{12}$$

$$A_{2y}' = -\mu_0(z' + vt' - b)H_{2x}'(t') \tag{13}$$

By taking the appropriate derivatives of Eqs. (12) and (13), one can compute the electric and magnetic fields in the slab system; and similarly, one can compute these fields in the laboratory system from Eqs. (1) and (2). These results lead to the equations

$$E_{0y}' = E_{0y} + vB_{0x} \qquad H_{0x}' = H_{0x} \tag{14}$$

$$E_{2y}' = E_{2y} + vB_{2x} \qquad H_{2x}' = H_{2x} \tag{15}$$

which are the correct transformation equations in the present approximation.[6]

We shall formulate the problem in the slab system, in which the diffusion equation has the same simple form as in Eq. (5), since Maxwell's equations are valid in the slab system. (In the laboratory system, the diffusion equation would have an extra term because of motional induction.[7]) Then, the equations for A_{1y}' are

$$(\partial^2 A_{1y}'/\partial z'^2) = \mu_0\sigma(\partial A_{1y}'/\partial t') \tag{16}$$

$$[\partial A_{1y}'/\partial z']_{z'=0} = -\mu_0 H_{0x}' \tag{17}$$

$$(b - l - vt')[\partial A_{1y}'/\partial z']_{z'=l} + A_{1y}'(l, t') = 0 \tag{18}$$

$$A_{1y}'(z', 0) = 0 \tag{19}$$

The boundary condition (18) is derived from the two equations expressing the continuity of A_{1y}' and of H_{1x}' at the surface $z' = l$, i.e.,

$$A_{1y}'(l, t') = -\mu_0(l + vt' - b)H_{2x}'(t') \tag{20}$$

$$[\partial A_{1y}'/\partial z']_{z'=l} = -\mu_0 H_{2x}'(t') \tag{21}$$

The system (16–19) can be solved by the Laplace transform method. Let s be the new independent variable and let a be the transform of A_{1y}'. We shall omit primes from now on. Then the transformed system is

$$(\partial^2 a/\partial z^2) = \mu_0\sigma s a \tag{22}$$

$$[\partial a/\partial z]_{z=0} = -\mu_0 H_0/s \tag{23}$$

$$(b - l)[\partial a/\partial z]_{z=l} + v[\partial^2 a/\partial z \partial s]_{z=l} + a(l, s) = 0 \tag{24}$$

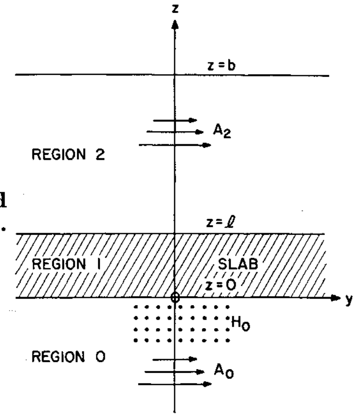

Fig. 1 Magnetic field and stationary slab.

The general solution of Eq. (22) is

$$a = \alpha \cosh z(\mu_0\sigma s)^{1/2} + \beta \sinh z(\mu_0\sigma s)^{1/2} \tag{25}$$

where α and β are constants to be determined by substitution of (25) into (23) and (24) and solving. A straightforward calculation leads to the result

$$a = s^{-3/2}\left(\frac{\mu_0}{\sigma}\right)^{1/2} H_0 \left\{\frac{[(\lambda/K) + (R/2\lambda)]\cosh\lambda[1 - (z/l)] + (\frac{1}{2}R + 1)\sinh\lambda[1 - (z/l)]}{[(\lambda/K) + (R/2\lambda)]\sinh\lambda + (\frac{1}{2}R + 1)\cosh\lambda}\right\} \tag{26}$$

where

$$K = l/(b - l) \qquad R = vl\mu_0\sigma \qquad \lambda = l(\mu_0\sigma s)^{1/2} = (\tau s)^{1/2} \tag{27}$$

The absolute value of R is the magnetic Reynolds number, according to Cowling.[4] The present R can, of course, have either a positive or a negative value, according to the direction of the motion.

4. Inversion of the Laplace Transformation

In order to transform back to the vector potential A_{1y}', we have to investigate the singularities of the function $a \exp(st)$, where a is given by (26). These singularities are located at $s = 0$ and at the zeros of the denominator of (26). Let us first investigate the singularity at $s = 0$. This can be done by expanding $a \exp(st)$ in a Laurent series in s. A somewhat lengthy but quite elementary calculation leads to the result that, for $R \neq -1$, the function $a \exp(st)$ has a second-order pole at $s = 0$. In the expansion, the coefficient of s^{-1} comes out to be

$$(H_0 t/\sigma l)[R/2(R + 1)] + \mu_0 H_0 l\{(1 + K^{-1} + \tfrac{3}{4}R)(R + 1)^{-1} - (z/l) + \tfrac{1}{4}R(R + 1)^{-1}(z^2/l^2) - \tfrac{1}{2}R(R + 1)^{-2}(K^{-1} + \tfrac{1}{3}R + \tfrac{1}{2})\} \tag{28}$$

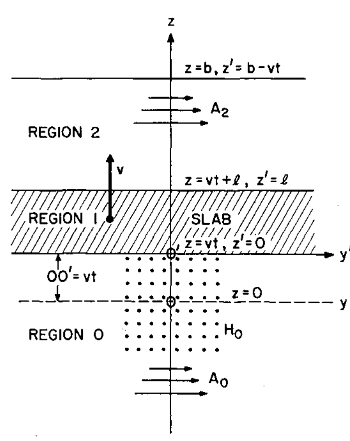

Fig. 2 Magnetic field and moving slab.

This expression is, therefore, the contribution to the inversion integral from the singularity at $s = 0$. It is not valid for $R = -1$. Accordingly, in making our calculations for the other singularities, we shall assume for the present that $R \neq -1$, leaving the case $R = -1$ until later.

The other singularities of $a \exp(st)$ occur at the roots of the equation

$$(\lambda/K) + (R/2\lambda) = -(\tfrac{1}{2}R + 1)\coth\lambda \qquad (29)$$

The evaluation of these roots is facilitated by the following theorems, whose proof is given in the Appendix: 1) for $R > -1$, all roots of (29) are imaginary; and 2) for $R < -1$, all roots are imaginary except for one real root and its negative. In the case of an imaginary root, one can replace λ by $i\zeta$, and then Eq. (29) is expressed in terms of real quantities:

$$(\zeta/K) - (R/2\zeta) = (\tfrac{1}{2}R + 1)\cot\zeta \qquad (30)$$

The contribution to the inversion integral from any one of the singularities (say at $s = s_n$) defined by (29) or (30) is calculated as follows. In $a \exp(st)$, the denominator of (26) is replaced by its derivative with respect to s, with the rest of (26) left unaltered; the resulting expression is then evaluated at $s = s_n$ and simplified with the aid of (29) or (30). Theorems from Laplace transform theory show that, if this procedure leads to a finite result, then the singularity at $s = s_n$ is a simple pole. All of the singularities discussed here turn out to be simple poles.

The calculation just described leads, for an imaginary root of (29), to the expression

$$[2\cos(\zeta_n z/l)/\zeta_n^2 L_n]e^{-\zeta_n^2(t/\tau)} \qquad (31)$$

where $\tau = \mu_0\sigma l^2$ and

$$L_n = -\frac{(\zeta_n^2 + \frac{1}{2}RK)K(\frac{1}{2}R + 1)}{(\zeta_n^2 - \frac{1}{2}RK)^2 + K^2(\frac{1}{2}R + 1)^2\zeta_n^2} - 1 \qquad (32)$$

For a real root, one can get the corresponding expressions by replacing ζ_n in (31) and (32) by $-i\lambda_n$.

Then, collecting results from (28) and (31), we have, for $R < -1$,

$$A_{1y} = \text{expression (28)} + 2\sum_{n=1}^{\infty}\frac{\cos(\zeta_n z/l)}{\zeta_n^2 L_n}e^{-\zeta_n^2(t/\tau)} + \frac{2\cosh(\lambda_0 z/l)}{\lambda_0^2 L_0}e^{\lambda_0^2(t/\tau)} \qquad (33)$$

All quantities in the preceding expression and in all equations to follow refer to the slab system, although the primes have been dropped for convenience.

In (33) the real root is denoted by λ_0, and

$$L_0 = \frac{(\lambda_0^2 - \frac{1}{2}RK)K(\frac{1}{2}R + 1)}{(\lambda_0^2 + \frac{1}{2}RK)^2 - K^2\lambda_0^2(\frac{1}{2}R + 1)^2} - 1 \qquad (34)$$

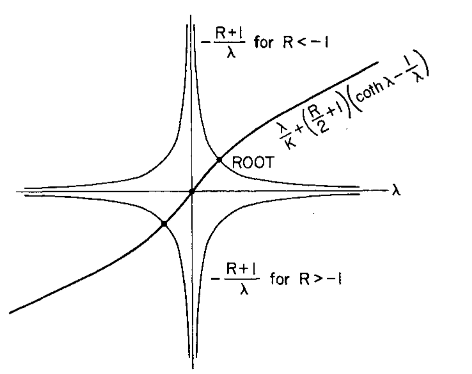

Fig. 3 Diagram illustrating lemmas on roots.

For $R > -1$, the expression for A_{1y} is the same as (33) but with the last term omitted, since for this case Eq. (29) has no real root.

Let us now consider the case $R = -1$. If we set R equal to -1 in (26), we can then calculate the expansion of $a \exp(st)$ near $s = 0$. The calculation shows that $a \exp(st)$ has a third-order pole at $s = 0$, and it leads, after a good deal of algebraic manipulation, to the following expression for the coefficient of s^{-1} in the expansion:

$$H_0\mu_0 l[\gamma + \beta(t/\tau) + \alpha(t/\tau)^2] \qquad (35)$$

where

$$\alpha = -3K/2(K + 6) \qquad (36)$$

$$\beta = \frac{3}{K + 6}\left[\frac{6K^2 + 60K + 120}{10(K + 6)} - \frac{K}{2}\left(\frac{z}{l}\right)^2\right] \qquad (37)$$

$$\gamma = \frac{3}{K + 6}\left\{\left(1 - \frac{z}{l}\right)^2\left[1 + \frac{K}{24}\left(3 + \frac{z}{l}\right)\left(1 - \frac{z}{l}\right)\right] + \frac{K}{280}\left(\frac{K + 14}{K + 6}\right) - \frac{1}{10}\left(\frac{K + 10}{K + 6}\right)\left[2 + \frac{K}{2}\left(1 - \frac{z^2}{l^2}\right) + \frac{K}{10}\left(\frac{K + 10}{K + 6}\right)\right]\right\} \qquad (38)$$

As to the other singularities of $a \exp(st)$ for the case $R = -1$, it will be shown in the Appendix that, for this case, Eq. (29) has a root at $\lambda = 0$, and all of its other roots are imaginary. The root at $\lambda = 0$ is already accounted for in the foregoing expansion of $a \exp(st)$ near $s = 0$, and therefore it only remains to consider the imaginary roots of (29). This is done in the same way as in the case $R \neq -1$ and with the same result, given by (31) and (32). Collecting all terms, we have, for $R = -1$,

$$A_{1y} = \text{expression (35)} + 2\sum_{n=1}^{\infty}\frac{\cos(\zeta_n z/l)}{\zeta_n^2 L_n}e^{-\zeta_n^2(t/\tau)} \qquad (39)$$

where L_n is given by (32) with $R = -1$.

It is of interest to ask what happens to the real root of Eq. (29) (with $R < -1$) as R is made to approach and then exceed -1. When this happens, the real root approaches and goes through the value zero and then becomes one of the imaginary roots occurring in the case $R > -1$. To show this, one expands $\coth\lambda$ in (29) and finds that, for λ near zero and R near -1, (29) reduces to the equation $\lambda^2 = -(R + 1)$ times a positive quantity. Meanwhile, the corresponding s value, initially positive, goes through zero and then becomes negative. Thus, the third-order pole occurring at $s = 0$ for $R = -1$ is the result of confluence of the simple pole arising from the real root of (29) with the second-order pole arising from the other factors in $a \exp(st)$.

5. Discussion of Results

We can best discuss the solution for the moving slab by comparison with the solution of the stationary problem, Eq. (7), which, as in all of the more usual diffusion problems, is the sum of a steady-state term representing a constant uniform magnetic field equal to H_0, and an infinite series of exponential decay terms representing transients. In the case of the moving slab, the solution is likewise the sum of a nonexponential expression and a series of exponential functions of time.

The nonexponential expression, (28) or (35), still represents the asymptotic state for the cases $R \geqq -1$, but now the magnetic field in the slab varies with z, as shown by the z^2 and higher terms in (28) and (35), and there is also an electric field in the slab. For $R \neq -1$, this electric field is constant and uniform according to the first term of (28), its magnitude reducing to $\frac{1}{2}vB_0$ and to $\frac{1}{2}H_0/\sigma l$ for $|R| \ll 1$ and $|R| \gg 1$, respectively. (Note that these field values are referred to the slab system.) For $R = -1$, the asymptotic electric field is nonuniform and increases indefinitely in time, according to (35) and (36).

The exponential terms in the solution are all decay terms if $R \geqq -1$, but for $R < -1$ there is one exponentially increasing term. Thus at $R = -1$ there is a marked change in the character of the solution. The physical reason for this seems to be that, for $R < -1$, magnetic transport outweighs magnetic diffusion, so that the field accumulates in the slab without limit, whereas, for $R > -1$, the field is drained away fast enough by diffusion so that no such accumulation occurs. This situation is described less mathematically in plasma literature, where the relative importance of transport vs diffusion is measured by the value of the magnetic Reynolds number R_m, which is arrived at through dimensional considerations and is considered to be always positive, no distinction being made between the two possible directions of motion. Thus the present treatment yields a quite precise mathematical interpretation of the magnetic Reynolds number.

Appendix

It is desired to prove the following theorems concerning the roots of Eq. (29): 1) for $R > -1$, all roots are imaginary; 2) for $R < -1$, all roots are imaginary except for one real root and its negative; and 3) for $R = -1$, zero is a root and all other roots are imaginary.

We first prove several lemmas, from which the theorems follow immediately.

Lemma 1

If $R < 0$, all roots are either real or imaginary. Let $\lambda = \alpha + i\beta$, and calculate the real and imaginary parts of Eq. (29). Divide one of these equations by the other; this yields the equation

$$\frac{\sinh 2\alpha}{2\alpha} = -\frac{\sin 2\beta}{2\beta}\left(\frac{\alpha^2 + \beta^2 + \frac{1}{2}RK}{\alpha^2 + \beta^2 - \frac{1}{2}RK}\right)$$

This equation cannot be valid if α and β are both nonvanishing; this is seen by taking absolute values and noting that the absolute value of the last factor is smaller than unity for $R < 0$.

Lemma 2

If $R > 0$, all roots are imaginary. Use the real part of Eq. (29), already calculated. It is satisfied by $\alpha = 0$. If $\alpha \neq 0$, divide by α; this leads to $(\sinh 2\alpha)/2\alpha$ = a negative quantity, which is impossible. Therefore, $\alpha = 0$.

Lemma 3, 4, and 5

If $-1 < R < 0$, Eq. (29) has no real roots; if $R < -1$, Eq. (29) has only one real root and its negative; and if $R = -1$, Eq. (29) has no real root except zero.

To prove these, write (29) in the form

$$(\lambda/K) + (\tfrac{1}{2}R + 1)[\coth\lambda - (1/\lambda)] = -(R + 1)/\lambda$$

Graph the right and left sides of this equation against λ, assuming λ real. It will be seen that for $-1 < R < 0$ there are no intersections; for $R < -1$ there is only one intersection and its negative; and for $R = 1$ there is only one intersection, at the origin (see Fig. 3).

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